

ON THE MONODROMY OF THE INFLECTION POINTS OF PLANE CURVES

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ABSTRACT. We prove that the monodromy group of the inflection points of plane curves of degree d is the symmetric group $\mathbb{S}_{3d(d-2)}$ for $d \geq 4$ and in the case $d = 3$ this group is the group of the projective transformations of \mathbb{P}^2 leaving invariant the nine inflection points of the Fermat curve of degree three.

0. INTRODUCTION.

Let $F(\bar{a}, \bar{z}) = \sum_{k+m+n=d} a_{k,m,n} z_1^k z_2^m z_3^n$ be the homogeneous polynomial of degree d in variables z_1, z_2, z_3 and of degree one in variables $a_{k,m,n}$, $k + m + n = d$. Denote by $\mathcal{C}_d \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$, where $K_d = \frac{d(d+3)}{2}$, the complete family of plane curves of degree d given by equation $F(\bar{a}, \bar{z}) = 0$. Let $\tilde{h}_d : \mathcal{C}_d \rightarrow \mathbb{P}^{K_d}$ and $h_d : \mathcal{I}_d = \mathcal{C}_d \cap \mathcal{H}_d \rightarrow \mathbb{P}^{K_d}$ be the restrictions of the projection $\text{pr}_1 : \mathbb{P}^{K_d} \times \mathbb{P}^2 \rightarrow \mathbb{P}^{K_d}$ to \mathcal{C}_d and \mathcal{I}_d respectively, where

$$\mathcal{H}_d = \{(\bar{a}, \bar{z}) \in \mathbb{P}^{K_d} \times \mathbb{P}^2 \mid \det\left(\frac{\partial^2 F(\bar{a}, \bar{z})}{\partial z_i \partial z_j}\right) = 0\}.$$

It is well-known (see, for example, [1]) that for a generic point $\bar{a}_0 \in \mathbb{P}^{K_d}$ the intersection of the curve $C_{\bar{a}_0} = \tilde{h}_d^{-1}(\bar{a}_0)$ and its Hessian curve $H_{C_{\bar{a}_0}}$ given by $\frac{\partial^2 F(\bar{a}_0, \bar{z})}{\partial z_i \partial z_j} = 0$ is the set of the inflection points of $C_{\bar{a}_0}$ containing $3d(d-2)$ points. Therefore for $d \geq 3$ we have $\deg h_d = 3d(d-2)$.

Let \mathcal{S}_d be the subvariety of \mathbb{P}^{K_d} consisting of the points \bar{a} such that the curves $C_{\bar{a}}$ are singular and let \mathcal{M}_d be the closure of subvariety of \mathbb{P}^{K_d} consisting of the points \bar{a} such that for $\bar{a} \in \mathcal{M}_d$ the curve $C_{\bar{a}}$ has a r -tuple inflection point with $r \geq 2$, i.e., $C_{\bar{a}}$ has a smooth point p such that the tangent line L to $C_{\bar{a}}$ at p and $C_{\bar{a}}$ have at p the intersection number $(L, C_{\bar{a}})_p = r + 2 \geq 4$. Let $\mathcal{B}_d = \mathcal{S}_d \cup \mathcal{M}_d$ (if $d = 3$ then $\mathcal{M}_3 = \emptyset$). Then $h_d : \mathcal{I}_d \setminus h_d^{-1}(\mathcal{B}_d) \rightarrow \mathbb{P}^{K_d} \setminus \mathcal{B}_d$ is an unramified covering and therefore it defines a homomorphism $h_{d*} : \pi_1(\mathbb{P}^{K_d} \setminus \mathcal{B}_d, \bar{a}_0) \rightarrow \mathbb{S}_{3d(d-2)}$ (here $\mathbb{S}_{3d(d-2)}$ is the symmetric group acting on the set $I_{\bar{a}} = C_{\bar{a}_0} \cap \mathcal{I}_d$). The group $\mathcal{G}_d = \text{Im } h_{d*}$ is called the *monodromy group of the inflection points of plane curves of degree d* .

The main result of the article is the following

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Theorem 1. *The group $\mathcal{G}_d = \mathbb{S}_{3d(d-2)}$ if $d \geq 4$ and \mathcal{G}_3 is a group of order 216 isomorphic to the group of the projective transformations of \mathbb{P}^2 leaving invariant the nine inflection points of the Fermat curve of degree three.*

The proof of Theorem 1 is given in Section 1. To prove Theorem 1, some properties of the variety \mathcal{I}_d near r -tuple inflection points of curves are investigated in this section. In Section 2, we investigate properties of \mathcal{I}_d near a node of a nodal curve of degree d which will be useful in the further investigations of the variety of the inflection points of plane curves of degree d .

1. PROOF OF THEOREM 1

1.1. On the monodromy of dominant morphisms. Let $\mathcal{B} \subset \mathbb{P}^K$ be a reduced hypersurface in the projective space \mathbb{P}^K . It is well-known that the fundamental group $\pi_1(\mathbb{P}^K \setminus \mathcal{B}, p)$ is generated by, so called, *bypasses* $\gamma_{q,L}$ around \mathcal{B} , that is, elements presented by loops $\Gamma_{q,L}$ of the following form. Let $L \subset \mathbb{P}^K$ be a germ of a smooth curve intersecting the curve \mathcal{B} at a point $q \in \mathcal{B}$, $L \not\subset \mathcal{B}$, and let $S^1 \subset L$ be a circle of small radius with center at q . The right orientation on \mathbb{P}^K , defined by complex structure, defines an orientation on S^1 and then $\Gamma_{q,L}$ is a loop consisting of a path l lying in $\mathbb{P}^K \setminus \mathcal{B}$ and connecting the point p with some point $q_1 \in S^1$, the loop S^1 (with right orientation) starting and ending at q_1 , and return to the point p along the path l .

Let $f : X \rightarrow \mathbb{P}^K$ be a dominant morphism, where X is a reduced variety, $\dim X = K$. Then there is a hypersurface $\mathcal{B} \subset \mathbb{P}^K$ (called the *discriminant locus* of f) such that $f : Y = X \setminus f^{-1}(\mathcal{B}) \rightarrow \mathbb{P}^K \setminus \mathcal{B}$ is an unramified finite covering. Note that Y is a smooth variety.

Let the degree of the covering $f : Y \rightarrow \mathbb{P}^K \setminus \mathcal{B}$ is equal to n . Then the covering f defines a homomorphism $f_* : \pi_1(\mathbb{P}^K \setminus \mathcal{B}, p) \rightarrow \mathbb{S}_n$ (called the *monodromy* of the covering f), where the image $G_f := f_*(\pi_1(\mathbb{P}^K \setminus \mathcal{B}, p))$ (called the *monodromy group* of f) is a subgroup of the symmetric group \mathbb{S}_n and it acts on the fibre $f^{-1}(p) = \{p_1, \dots, p_n\}$ as follows. A loop $\Gamma \subset \mathbb{P}^K \setminus \mathcal{B}$ representing an element $\gamma \in \pi_1(\mathbb{P}^K \setminus \mathcal{B}, p)$ can be lifted to Y and this lift consists of n paths $\Gamma_1, \dots, \Gamma_n \subset Y$ starting and ending at the points of $f^{-1}(p)$. Therefore this lift defines an action $f_*(\gamma)$ on $f^{-1}(p)$ which sends the start point $p_i \in \Gamma_i$ to the end point of Γ_i for each $i = 1, \dots, n$.

The following Lemma is obvious.

Lemma 1. *In notations used above, let*

- (1) $\nu : \tilde{L} \rightarrow f^{-1}(L)$ *be the normalization of the curve $f^{-1}(L)$, where $L \subset \mathbb{P}^N$;*
- (2) *the preimage $\tilde{f}^{-1}(q)$ consist of k points q_1, \dots, q_k , where $\tilde{f} = f \circ \nu$ and $q \in L \cap \mathcal{B}$;*
- (3) n_i *be the ramification index of \tilde{f} at q_i , $n_1 + \dots + n_k = n$.*

Then $f_(\gamma_{q,L}) = c_1 \cdot \dots \cdot c_k \in \mathbb{S}_n$ is the product of k pairwise disjoint cycles c_i of length n_i .*

Let $V \subset \mathbb{P}^K$ be a small neighbourhood (in complex-analytic topology) of a point $q \in \mathcal{B}$ bi-holomorphic to a polidisk

$$\Delta^K = \{(z_1, \dots, z_K) \in \mathbb{C}^K \mid |z_j| < \varepsilon \ll 1 \text{ for } j = 1, \dots, K\}$$

with center at q . The imbedding $i : V \setminus \mathcal{B} \hookrightarrow \mathbb{P}^K \setminus \mathcal{B}$ induces a homomorphism $i_* : \pi_1(V \setminus \mathcal{B}) \rightarrow \pi_1(\mathbb{P}^K \setminus \mathcal{B})$ and a homomorphism $f_{*,loc} = f_* \circ i_* : \pi_1(V \setminus \mathcal{B}) \rightarrow G_f$ defined by i_* and f_* uniquely up to conjugation in G . If a neighbourhood V is small enough then the image $G_{f,q} := f_{*,loc}(\pi_1(V \setminus \mathcal{B}))$ does not depend of V and it is called the *local monodromy group* of f at the point q . The following Claim is well-known.

Claim 1. *In notations used above, let q be a smooth point of \mathcal{B} and a curve L intersects with \mathcal{B} transversally at q . Then the local monodromy group $G_{f,q}$ is cyclic and it is generated by $f_*(\gamma_{q,L})$.*

Lemma 2. *Let $Z \subset \Delta^{K+1}$ be a germ of a reduced complex-analytic variety, $\dim Z = K$, in the polidisk $\Delta^{K+1} = \{(z_1, \dots, z_{K+1}) \in \mathbb{C}^{K+1} \mid |z_j| < \varepsilon \ll 1 \text{ for } j = 1, \dots, n+1\}$, $o = (0, \dots, 0) \in Z$, and let the restriction $f : Z \rightarrow \Delta^K$ of the projection $pr : \Delta^{K+1} \rightarrow \Delta^K$, $pr : (z_1, \dots, z_{K+1}) \mapsto (z_1, \dots, z_K)$, has the following properties:*

- (i) *f is a proper finite holomorphic map, $\deg f = n$;*
- (ii) *the discriminant locus $\mathcal{B} \subset \Delta^K$ is a smooth hypersurface;*
- (iii) *the local degree $\deg_o f = n$;*
- (iv) *there is a germ $L \subset \Delta^K$, $\dim L = 1$, such that L meets \mathcal{B} at the point $o' = f(o)$, $L \not\subset \mathcal{B}$, and $E = f^{-1}(L)$ is a smooth curve.*

Then o is a non-singular point of Z .

Proof. Without loss of generality, we can assume that \mathcal{B} is given by $z_1 = 0$ and, by Weiershtrass preparation theorem, Z is given by equation of the form

$$z_{K+1}^n + \sum_{j=0}^{n-1} \alpha_j(z_1, \dots, z_K) z_{K+1}^j = 0, \quad (1)$$

where $\alpha_j(z_1, \dots, z_K) \in \mathbb{C}[[z_1, \dots, z_K]]$. By Claim 1, $\alpha_j(0, z_2, \dots, z_K) = 0$ for each $j = 0, \dots, n-1$. Therefore z_1 is a divisor of each power series $\alpha_j(z_1, \dots, z_K)$, $\alpha_j(z_1, \dots, z_K) = z_1^{k_j} \beta_j(z_1, \dots, z_K)$, where $\beta_j(z_1, \dots, z_K)$ is a power series coprime with z_1 and k_j is a positive integer. Let the germ L be given by parametrization

$$z_j = \sum_{l=1}^{\infty} c_{j,l} t^l, \quad j = 1, \dots, K. \quad (2)$$

Then the curve $E = f^{-1}(L)$ is given by (1) and (2). Therefore $\beta_0(0, \dots, 0) \neq 0$, $c_{1,1} \neq 0$, and $k_0 = 1$, since E is a smooth curve at o . Now, the smoothness of Z follows from inequality $\beta_0(0, \dots, 0) \neq 0$. \square

1.2. On r -tuple inflection points. In that follows we shall use the following well-known properties of plane curves of degree $d \geq 3$ (see, for example [1]). First of all remind that the Hessian curve H_C of a plane curve C is independent on the choice of coordinates; H_C intersects C at the singular and inflection points of C if C does not contain a line as its irreducible component. If a line L is a component of C , then L also is a component of H_C . Moreover, if \bar{z}_0 is a r -tuple inflection point of C , then (Theorem 1 in subsection 7.3 in [1]¹) the intersection number $(C, H_C)_{\bar{z}_0}$ at the point \bar{z}_0 is equal to r . Therefore we have

Claim 2. *Let $C_{\bar{a}_0} = C \cup (\cup_{j=1}^k L_j)$ be the union of a curve C of degree $\deg C \geq 3$ and k lines L_j (may be, $k = 0$). Let $\{\bar{z}_1, \dots, \bar{z}_n\}$ be the set of the inflection points of $C_{\bar{a}_0}$ which do not lie in $\cup_{j=1}^k L_j$. Then there is a small (in complex analytic topology) neighbourhood $\mathcal{U} \subset \mathbb{P}^{K_d}$ of the point \bar{a}_0 such that $h_d^{-1}(\mathcal{U})$ is the disjoint union of $n+1$ open sets V_l , $l = 1, \dots, n+1$, such that $(\bar{a}_0, \bar{z}_l) \in V_l$ for $l \leq n$. The local degree $\deg_{(\bar{a}_0, \bar{z}_l)} h_d$ of the covering h_d at a point (\bar{a}_0, \bar{z}_l) is equal to r_l if \bar{z}_l is a r_l -tuple inflection point of $C_{\bar{a}_0}$. In particular, if \bar{z}_l is a simple inflection point (that is, $r_l = 1$) and \mathcal{U} is chosen small enough, then $h_d|_{V_l} : V_l \rightarrow \mathcal{U} = h_d(V_l)$ is a bi-holomorphic map.*

Proof. The local degree $\deg_{(\bar{a}_0, \bar{z}_0)} h_d$ of the covering h_d at the point (\bar{a}_0, \bar{z}_0) is equal to the intersection number $(\mathcal{I}, \text{pr}_1^{-1}(\bar{a}_0))_{(\bar{a}_0, \bar{z}_0)}$ of the variety \mathcal{I} and the fibre $\text{pr}_1^{-1}(\bar{a}_0)$ at (\bar{a}_0, \bar{z}_0) . On the other hand, $(\mathcal{I}, \text{pr}_1^{-1}(\bar{a}_0))_{(\bar{a}_0, \bar{z}_0)}$ is equal to the intersection number of $C_{\bar{a}_0}$ and its Hessian $H_{C_{\bar{a}_0}}$ at \bar{z}_0 in \mathbb{P}^2 . \square

The following Proposition is well-known, but since I do not know a good reference, a proof will be given.

Proposition 1. *The variety \mathcal{M}_d is an irreducible hypersurface in \mathbb{P}^{K_d} for each $d \geq 4$. There is a non-empty Zariski open neighbourhood $\mathfrak{M}_d \subset \mathcal{M}_d$ such that for each $\bar{a} \in \mathfrak{M}_d$ the curve $C_{\bar{a}}$ is non-singular and it has $3d(d-2) - 1$ inflection points (that is, it has the only one multiple ($r = 2$) inflection point).*

Proof. Denote by $\mathcal{D}_r \subset \mathcal{C}_d$, $r = 2, 3$, the subfamilies of curves given by

$$z_1 S(z-1, z_2, z_3) + z_2^{r+2} R(z_2, z_3) = 0, \quad (3)$$

where $S(z_1, z_2, z_3)$ is the generic homogeneous polynomial of degree $d-1$ in variables z_1, z_2, z_3 and $R(z_2, z_3)$ is the generic homogeneous polynomial of degree $d-(r+2)$ in variables z_2, z_3 . Denote also by $D_r = \text{pr}_1(\mathcal{D}_r) \subset \mathbb{P}^{K_d}$ the image of \mathcal{D}_r under the projection pr_1 . Obviously, $D_3 \subset D_2$, D_2 and D_3 are irreducible projective varieties, and it is easy to see that $\dim D_2 = \frac{(d-1)(d+2)}{2} + (d-3) = K_d - 4$ and $\dim D_3 = \frac{(d-1)(d+2)}{2} + (d-4) = K_d - 5$.

¹ In [1], Theorem 1 is proved under the additional assumption that there is not a line among the irreducible components of C . But, it is easy to see that this assumption is not used in the proof of this Theorem.

Similarly, let $\mathcal{D}_{2,2} \subset \mathcal{D}_2$ be the subfamily of curves given by

$$z_1 z_3 S(z_1, z_2, z_3) + z_2^4 (R_1(z_1, z_2) + R_2(z_2, z_3)) = 0, \quad (4)$$

where $\deg S(z_1, z_2, z_3) = d-2$ and $\deg R_1(z_1, z_2) = \deg R_2(z_2, z_3) = d-4$; $\mathcal{D}_{2,2,1} \subset \mathcal{D}_2$ the subfamily of curves given by

$$z_1 z_3 S(z_1, z_2, z_3) + z_2^4 z_3 R_1(z_2, z_3) + z_1^4 R_2(z_1, z_2) = 0, \quad (5)$$

where $\deg S(z_1, z_2, z_3) = d-2$, $\deg R_1(z_1, z_2) = d-5$, and $\deg R_2(z_2, z_3) = d-4$; and $\mathcal{D}_{2,2,2} \subset \mathcal{D}_2$ the subfamily of curves given by

$$z_1 S(z_1, z_2, z_3) + z_2^4 z_3^4 R_1(z_2, z_3) = 0, \quad (6)$$

where $\deg S(z_1, z_2, z_3) = d-1$ and $\deg R_1(z_2, z_3) = d-8$. As above, denote by $D_{2,2} = \text{pr}_1(\mathcal{D}_{2,2})$, $D_{2,2,1} = \text{pr}_1(\mathcal{D}_{2,2,1})$, and $D_{2,2,2} = \text{pr}_1(\mathcal{D}_{2,2,2})$ the images respectively of $\mathcal{D}_{2,2}$, $\mathcal{D}_{2,2,1}$, and $\mathcal{D}_{2,2,2}$ under the projection pr_1 . Obviously, $D_{2,2}$, $D_{2,2,1}$, and $D_{2,2,2}$ are irreducible projective varieties, and it is easy to see that $\dim D_{2,2} = K_d - 7$ and $\dim D_{2,2,1} = \dim D_{2,2,2} = K_d - 8$.

Let p_1 be a r -tuple point, $r = 2$ or $r \geq 3$, of a curve C . We choose homogeneous coordinates (z_1, z_2, z_3) so that the point p_1 has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to C at the point p_1 . Then an equation of C has the form (3).

Let C has two 2-tuple inflection points p_1 and p_2 , and let L_1 and L_2 be the tangent lines to C respectively at p_1 and p_2 . We have three possibilities depending on the position of the points p_1 and p_2 with respect to the curve C : either $p_1 \notin L_2$ and $p_2 \notin L_1$, or $p_2 \in L_1$, but $L_1 \neq L_2$, or $L_1 = L_2$. In the first case, if we choose homogeneous coordinates (z_1, z_2, z_3) so that the point p_1 has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to C at the point p_1 , the point p_2 has coordinates $(1, 0, 0)$ and the line $L_2 = \{z_3 = 0\}$ is the tangent line to C at the point p_2 , then an equation of C has the form (4). In the second case, if we choose homogeneous coordinates (z_1, z_2, z_3) so that the point p_1 has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to C at the point p_1 , the point p_2 has coordinates $(0, 1, 0)$ and the line $L_2 = \{z_3 = 0\}$ is the tangent line to C at the point p_2 , then an equation of C has the form (5). In the third case, if we choose homogeneous coordinates (z_1, z_2, z_3) so that the point p_1 has coordinates $(0, 0, 1)$, the point p_2 has coordinates $(0, 1, 0)$, and the line $L_1 = \{z_1 = 0\}$ is the tangent line to C at the points p_1 and p_2 , then an equation of C has the form (6).

The group $PGL(3, \mathbb{C})$ acts on \mathbb{P}^{K_d} so that $g(\bar{a}) \in \mathcal{M}_d$ for each $g \in PGL(3, \mathbb{C})$ and $\bar{a} \in D_2$. Denote by $\Gamma_{\dagger} \subset PGL(3, \mathbb{C})$ the subgroup leaving invariant the variety D_{\dagger} , where \dagger is either 2, or 3, or $\{2, 2\}$, or $\{2, 2, 1\}$, or $\{2, 2, 2\}$.

Obviously, the groups Γ_r , $r = 2$ or $r \geq 3$, contain a subgroup Γ_0 consisting of the elements of the following form

$$g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \quad (7)$$

and it is easy to see that Γ_0 is a subgroup of finite index in Γ_r , since each non-singular curve C can have only finitely many multiple inflection points. Therefore $\dim \Gamma_r = 5$. Note that if there exists a curve $C_{\bar{a}}$, $\bar{a} \in \mathcal{M}_d$, which has only one multiple inflection point, then $\Gamma_0 = \Gamma_2$.

Similarly, the diagonal group Δ is a subgroup of $\Gamma_{2,2}$ of finite index; the group consisting of the elements of the form

$$g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ 0 & 0 & g_{3,3} \end{pmatrix} \in PGL(3, \mathbb{C}) \quad (8)$$

is a subgroup of $\Gamma_{2,2,1}$ of finite index; and the group consisting of the elements of the form

$$g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ g_{3,1} & 0 & g_{3,3} \end{pmatrix} \in PGL(3, \mathbb{C}) \quad (9)$$

is also a subgroup of $\Gamma_{2,2,1}$ of finite index. Therefore $\dim \Gamma_{2,2} = 2$, $\dim \Gamma_{2,2,1} = 3$, and $\dim \Gamma_{2,2,2} = 4$.

Consider the morphism $\nu : PGL(3, \mathbb{C}) \times D_2 \rightarrow \mathbb{P}^{K_d}$ given by $\nu((g, \bar{a})) = g(\bar{a})$. Obviously, the image $\text{Im } \nu \subset \mathcal{M}_d$ is an everywhere dense subset of \mathcal{M}_d . The preimage of a point $\nu((g_0, \bar{a}_0))$, where \bar{a}_0 is a generic point of D_2 , is the variety $\{(g, g^{-1}g_0(\bar{a}_0)) \mid g \in \Gamma_r\}$ of dimension five. Therefore

$$\dim \nu(PGL(3, \mathbb{C}) \times D_r) = \dim PGL(3, \mathbb{C}) + \dim D_r - \dim \Gamma_r = K_d - r + 1.$$

In particular, $\dim \mathcal{M}_d = \dim \nu(D_2) = K_d - 1$ and therefore \mathcal{M}_d is a hypersurface in \mathbb{P}^{K_d} .

Similar calculations give rise $\dim \nu(PGL(3, \mathbb{C}) \times D_{2,2}) = K_d - 2$,

$$\dim \nu(PGL(3, \mathbb{C}) \times D_{2,2,1}) = K_d - 3, \quad \dim \nu(PGL(3, \mathbb{C}) \times D_{2,2,2}) = K_d - 4.$$

It is well-known that \mathcal{S}_d is a divisor in \mathbb{P}^{K_d} and it is easy to see that $\mathcal{M}_d \not\subset \mathcal{S}_d$. Therefore $\dim \mathcal{M}_d \cap \mathcal{S}_d = K_d - 2$. Now,

$$\mathfrak{M}_d = \mathcal{M}_d \setminus (\overline{\nu(PGL(3, \mathbb{C}) \times (D_{2,2} \cup D_{2,2,1} \cup D_{2,2,2}))} \cup \mathcal{S}_d)$$

is the desired variety, where by \overline{M} is denoted the closure of a variety $M \subset \mathbb{P}^{K_d}$.

Therefore $\Gamma_2 = \Gamma_0$ and hence $\nu(PGL(3, \mathbb{C}) \times D_2)$ is irreducible, since there is a point $\bar{a} \in D_2$ such that $C_{\bar{a}}$ is non-singular and it has only one multiple inflection point (more precisely, 2-tuple inflection point). \square

1.3. On a quasi-imbedding of the permutation group \mathcal{G}_d into \mathcal{G}_{d+1} . Denote by (G, n) a subgroup G of the symmetric group \mathbb{S}_n acting on a set J_n of cardinality n as permutations and call (G, n) a *permutation group*.

Let $J_n = J_m \sqcup J_k$, $m + k = n$, be a partition of J_n and (G, n) a permutation group such that the action of G on J_n leaves invariant the set J_m . The action of G on J_m defines a homomorphism $\varphi_{n,m} : G \rightarrow \mathbb{S}_m$, that is, it defines the permutation group (G_{J_m}, m) , where $G_{J_m} = \text{Im } \varphi_{n,m}$. We say that a permutation group (H, m) is *quasi-imbedded* in a permutation group (G, n) (and denote this quasi-imbedding by $(H, m) \prec (G, n)$) if

- (i) $n \geq m$ and there is a partition $J_n = J_m \sqcup J_{n-m}$ such that G leaves invariant the set J_m ;
- (ii) the permutation groups (G_{J_m}, m) and (H, m) are isomorphic as permutation groups.

Remind that the group \mathcal{G}_d is the image of $\pi_1(\mathbb{P}^{K_d} \setminus \mathcal{B}_d, \bar{a}_0)$ under the homomorphism $h_{d*} : \pi_1(\mathbb{P}^{K_d} \setminus \mathcal{B}_d, \bar{a}_0) \rightarrow \mathbb{S}_{3d(d-2)}$, where the symmetric group $\mathbb{S}_{3d(d-2)}$ acts on $J_{3d(d-2)} := h_d^{-1}(\bar{a}_0)$. Therefore in what follows, the group \mathcal{G}_d will be considered as a permutation group $(\mathcal{G}_d, 3d(d-2))$, but it will be denoted again simply by \mathcal{G}_d .

Claim 3. *For each $d \geq 3$ there is a quasi-imbedding of \mathcal{G}_d in \mathcal{G}_{d+1} .*

Proof. For each $g \in \mathcal{G}_d$ let us choose a continuous loop $\Gamma : [0, 1] \rightarrow \mathbb{P}^{K_d} \setminus \mathcal{B}_d$ representing an element $\gamma \in h_{d*}^{-1}(g) \subset \pi_1(\mathbb{P}^{K_d} \setminus \mathcal{B}_d, \bar{a}_0)$, where $[a, b] = \{t \in \mathbb{R} \mid a \leq t \leq b\}$ is a segment in \mathbb{R} . Then the action of g on $J_{3d(d-2)}$ is defined by the disjoint union $h_d^{-1}(\Gamma([0, 1])) = \bigsqcup_{j=1}^{3d(d-2)} l_j([0, 1])$ of $3d(d-2)$ continuous paths $l_j : [0, 1] \rightarrow \mathbb{P}^{K_{d+1}} \setminus \mathcal{B}_{d+1}$ starting and ending at the points of $h_d^{-1}(\bar{a}_0)$. Denote $\bar{z}_{\tau,j} = \text{pr}_2(l_j(\tau))$.

Since $h_d^{-1}(\Gamma([0, 1]))$ is one-dimensional as a topological space, we can choose a line $L \subset \mathbb{P}^2$ such that $L \cap \text{pr}_2(\bigsqcup_{j=1}^{3d(d-2)} l_j([0, 1])) = \emptyset$. For each $\bar{a} \in \mathbb{P}^{K_d}$ the curve $C_{\bar{a}} \cup L$ has degree $d+1$. Therefore the choice of L defines an imbedding $\lambda_d : \mathbb{P}^{K_d} \hookrightarrow \mathbb{P}^{K_{d+1}}$ given by $\lambda_d : \bar{a} \mapsto \bar{b} = \tilde{h}_{d+1}(C_{\bar{a}} \cup L) \in \mathbb{P}^{K_{d+1}}$ for $\bar{a} \in \mathbb{P}^{K_d}$.

Consider the loop $\lambda_d(\Gamma([0, 1]))$. By Claim 2, for each $\tau \in [0, 1]$ there is a small (in complex analytic topology) connected neighbourhood $\mathcal{U}_\tau \subset \mathbb{P}^{K_{d+1}}$ of the point $\lambda_d(\Gamma(\tau))$ such that $h_{d+1}^{-1}(U_\tau)$ is the disjoint union of $3d(d-2) + 1$ open sets $V_{\tau,j}$, $j = 1, \dots, 3d(d-2) + 1$, such that $(\lambda_d(\Gamma(\tau)), \bar{z}_{\tau,j}) \in V_{\tau,j}$ and $h_{d+1} : V_{\tau,j} \rightarrow U_\tau$ is a bi-holomorphic map for $j \leq 3d(d-2)$, where $\{\bar{z}_{\tau,1}, \dots, \bar{z}_{\tau,3d(d-2)}\}$ is the set of the inflection points of the curve $C_{\Gamma(\tau)}$. Note that we can choose U_0 and U_1 such that $U_0 = U_1$ and this open set does not depend on $g \in \mathcal{G}_d$.

Let $\Delta_\tau = \{t \in [0, 1] \mid \tau - \varepsilon_\tau < t < \tau + \varepsilon_\tau\}$ be segments in $[0, 1]$ such that $\lambda_d(\Gamma(\Delta_\tau)) \subset U_\tau$ for $0 < \tau < 1$ and, similarly, $\Delta_0 = \{t \in [0, 1] \mid t < \varepsilon_0\}$ and $\Delta_1 = \{t \in [0, 1] \mid t > 1 - \varepsilon_1\}$ be segments such that $\lambda_d(\Gamma(\Delta_0)) \subset U_0$ and $\lambda_d(\Gamma(\Delta_1)) \subset U_1$.

Consider a path $\Theta : [0, 1] \rightarrow \mathbb{P}^{K_{d+1}} \times [0, 1]$ given by $\Theta : \tau \mapsto (\lambda(\Gamma(\tau)), \tau)$. Obviously, $\{U_\tau \times \Delta_\tau\}_{\tau \in [0,1]}$ is a cover of the path $\Theta([0, 1])$. Since $\Theta([0, 1])$ is a compact, we can

choose a finite cover

$$\{U_0 \times \Delta_0, U_{\tau_1} \times \Delta_{\tau_1}, \dots, U_{\tau_n} \times \Delta_{\tau_n}, U_1 \times \Delta_1\}, \quad 0 = \tau_0 < \tau_1 < \dots < \tau_n < \tau_{n+1} = 1.$$

It is easy to see that $U_{\tau_j} \cap U_{\tau_{j+1}} \neq \emptyset$ for each $j = 0, \dots, n$. Let us choose a point $\bar{b}_0 \in U_0 \setminus \mathcal{B}_{d+1}$ and points $\bar{b}_{j,j+1} \in (U_{\tau_j} \cap U_{\tau_{j+1}}) \setminus \mathcal{B}_{d+1}$ for $j = 0, \dots, n$. Each variety $U_{\tau_j} \setminus \mathcal{B}_{d+1}$ is connected. Therefore for $0 \leq j \leq n+1$ we can connect the point $\bar{b}_{j-1,j}$ with $\bar{b}_{j,j+1}$ by a continuous path $\Gamma_j \subset U_{\tau_j} \setminus \mathcal{B}_{d+1}$, where $\bar{b}_{-1,0} = \bar{b}_{n+1,n+2} = \bar{b}_0$.

Consider the set $J_{3(d^2-1)} = h_{d+1}^{-1}(\bar{b}_0) = \{\tilde{q}_1, \dots, \tilde{q}_{3d(d-1)}, \dots, \tilde{q}_{3(d^2-1)}\}$, where $\tilde{q}_j \in V_{0,j}$ for $j = 1, \dots, 3d(d-1)$. Denote $\tilde{J}_{3d(d-2)} = \{\tilde{q}_1, \dots, \tilde{q}_{3d(d-2)}\} \subset J_{3(d^2-1)}$. The consecutive join of the paths Γ_j , $j = 0, \dots, n+1$, is a continuous loop

$$\tilde{\Gamma} = \Gamma_0 \cup \dots \cup \Gamma_{n+1} \subset \mathbb{P}^{K_{d+1}} \setminus \mathcal{B}_{d+1}$$

starting and ending at \bar{b}_0 . Then the start and end points of the paths \tilde{l}_j , entering in the disjoint union $h_{d+1}^{-1}(\tilde{\Gamma}) = \bigsqcup_{j=1}^{3(d^2-1)} \tilde{l}_j$ of $3(d^2-1)$ continuous paths, are contained in $J_{3(d^2-1)}$. Let $\tilde{g} = h_{d+1*}(\tilde{\gamma}) \in \mathcal{G}_{d+1}$, where $\tilde{\gamma} \in \pi_1(\mathbb{P}^{K_{d+1}} \setminus \mathcal{B}_{d+1}, \bar{b}_0)$ is represented by $\tilde{\Gamma}$. If we number the paths \tilde{l}_j so that the start point of \tilde{l}_j is \tilde{q}_j for $j \leq 3d(d-2)$, then it easily follows from the construction of $\tilde{\Gamma}$ that

- (1) the end point of \tilde{l}_j lies also in $\tilde{J}_{3d(d-2)}$ for each $j \leq 3d(d-2)$;
- (2) \tilde{g} leaves invariant the set $\tilde{J}_{3d(d-2)}$;
- (3) the restriction of the action of \tilde{g} to $\tilde{J}_{3d(d-2)}$ is the same as the action of g on $J_{3d(d-2)}$ if we identify $\tilde{J}_{3d(d-2)}$ with $J_{3d(d-2)}$.

Denote by $\tilde{\mathcal{G}}_d$ a permutation subgroup of \mathcal{G}_{d+1} generated by the elements \tilde{g} , where $g \in \mathcal{G}_d$. Obviously, the permutation subgroup $\tilde{\mathcal{G}}_d$ defines a quasi-imbedding of \mathcal{G}_d in \mathcal{G}_{d+1} . \square

1.4. Behaviour of the covering h_d near the inflection points of the Fermat curve. Let $F_d \subset \mathbb{P}^2$ be the Fermat curve of degree d , i.e., the curve given by equation $z_1^d + z_2^d + z_3^d = 0$. It has $3d$ the $(d-2)$ -tuple inflection points $\bar{z}_{j,l}$, where

$$\bar{z}_{1,l} = (0, \mu_l, 1), \quad \bar{z}_{2,l} = (\mu_l, 0, 1), \quad \bar{z}_{3,l} = (\mu_l, 1, 0), \quad l = 1, \dots, d, \quad \mu_l = e^{\pi i(2l-1)/d}.$$

The subgroup G_d of $PGL(3, \mathbb{C})$, generated by

$$g_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, g_3 = \begin{pmatrix} e^{2\pi i/d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

leaves invariant the curve F_d and acts transitively on the set of its inflection points. As a group, it is isomorphic to $\mathbb{Z}_d^2 \rtimes \mathbb{S}_3$.

Let $f_d \in \mathbb{P}^{K_d}$ be the point corresponding to the Fermat curve F_d and let $\mathbb{C}^{K_d} \subset \mathbb{P}^{K_d}$ be the affine space given by $a_{0,0,d} \neq 0$. Denote again the non-homogeneous coordinates in \mathbb{C}^{K_d} by $a_{k,m,n}$, $(k, m, n) \neq (0, 0, d)$ (here we assume that $a_{0,0,d} = 1$). Then the point

$f_d \in \mathbb{C}^{K_d}$ has the following coordinates: $a_{d,0,0} = a_{0,d,0} = 1$, and all other coordinates are equal to zero.

Consider a neighbourhood

$$U_\varepsilon = \{\bar{a} \in \mathbb{C}^{K_d} \mid \begin{array}{l} |a_{k,m,n} - 1| < \varepsilon, \text{ for } (k,m,n) = (d,0,0) \text{ or } (0,d,0) \text{ and} \\ |a_{k,m,n}| < \varepsilon \text{ for all } (k,m,n) \neq (d,0,0) \text{ or } (0,d,0) \end{array}\}$$

of the point f_d .

Claim 4. *For positive $\varepsilon \ll 1$ the variety $\mathcal{U}_\varepsilon = h_d^{-1}(U_\varepsilon)$ is the disjoint union of $3d$ irreducible varieties $\mathcal{U}_{j,l}$, where for each $j = 1, 2, 3$ and $l = 1, \dots, d$ the variety $\mathcal{U}_{j,l}$ is defined by the following condition: the $(d-2)$ -tuple inflection point $\bar{z}_{j,l}$ of the curve F_d lies in $\mathcal{U}_{j,l}$. The restriction $\mathcal{U}_{j,l} \rightarrow U_\varepsilon$ of the morphism h_d to each $\mathcal{U}_{j,l}$ has degree $d-2$.*

Proof. Obviously, Claim 4 is true in the case $d = 3$. Therefore we will assume that $d \geq 4$.

It is easy to see that if ε is small enough, then the variety $\mathcal{U}_\varepsilon = h_d^{-1}(U_\varepsilon) = \bigsqcup_{j=1}^3 \bigsqcup_{l=1}^d \mathcal{U}_{j,l}$ is the disjoint union of $3d$ varieties $\mathcal{U}_{j,l}$ such that $(f_d, \bar{z}_{j,l}) \in \mathcal{U}_{j,l}$. Therefore to prove Claim 4, it suffices to prove that $\mathcal{U}_{1,1}$ is an irreducible variety, since the induced actions of the group G_d on $\mathbb{P}^{K_d} \times \mathbb{P}^2$ and \mathbb{P}^{K_d} leave invariant the varieties U_ε , $\tilde{h}_d^{-1}(U_\varepsilon)$, and \mathcal{U}_ε , $g \circ h_d = h_d \circ g$ for each $g \in G_d$, and this action induces a transitive action on the set of varieties $\mathcal{U}_{j,l}$. Obviously, the restriction of h_d to each $\mathcal{U}_{j,l}$ has degree $d-2$.

To prove that $\mathcal{U}_{1,1}$ is an irreducible variety, consider a family C_u of curves in $\mathbb{P}^{K_d} \times \mathbb{P}^2$ given by

$$F(u, \bar{z}) := z_1^d + z_2^d + z_3^d + uz_1^2 z_3^{d-2} = 0 \quad (10)$$

and its image $L = \tilde{h}(C_u) \simeq \mathbb{C}$. Denote by $L_\varepsilon = L \cap U_\varepsilon$. The family C_u lies in $L \times \mathbb{P}^2 \subset \mathbb{P}_{K_d} \times \mathbb{P}^2$. It is easy to check that in coordinates $(u; z_1, z_2, z_3)$ the Hessian of the family C_u is

$$\mathcal{H}(u; \bar{z}) = \det \left(\frac{\partial^2 F(u, \bar{z})}{\partial z_i \partial z_j} \right) = d(d-1)z_2^{d-2}z_3^{d-4}H(u, z_1, z_3), \quad (11)$$

where

$$\begin{aligned} H(u, z_1, z_3) = & (d(d-1)z_1^{d-2} + 2uz_3^{d-2})(d(d-1)z_3^2 + (d-2)(d-3)uz_1^2) - 4(d-2)^2u^2z_1^2z_3^{d-2} = \\ & \frac{d!}{(d-4)!}uz_1^d + d^2(d-1)^2z_1^{d-2}z_3^2 - 2(d-1)(d-2)u^2z_1^2z_3^{d-2} + 2d(d-1)uz_3^d. \end{aligned}$$

Therefore the curve $J = h_d^{-1}(L) \subset \mathbb{C} \times \mathbb{P}^2$, given by $F(u, \bar{z}) = \mathcal{H}(u, \bar{z}) = 0$, is the union of curves, $J = J_1 \cup J_2 \cup J_3$ (if $d = 4$ then $J_3 = \emptyset$), where J_2 and J_3 (if $d \geq 5$) are the intersections of the surface given by (10) and, respectively, two surfaces given

by $z_2 = 0$ and $z_3 = 0$, and J_1 is the intersection of the surface given by equation (10) and the surface \overline{H}_1 given by

$$\frac{d!}{(d-4)!}uz_1^d + d^2(d-1)^2z_1^{d-2}z_3^2 - 2(d-1)(d-2)u^2z_1^2z_3^{d-2} + 2d(d-1)uz_3^d = 0. \quad (12)$$

It is easy to see that $(\cup_{l=1}^d \mathcal{U}_{1,l}) \cap (J_2 \cup J_3) = \emptyset$ and $(\cup_{l=1}^d \mathcal{U}_{1,l}) \cap J_1 \subset \mathbb{C} \times \mathbb{C}^2$, where $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{z_3 = 0\}$. Let $x = z_1/z_3$, $y = z_2/z_3$ be coordinates in \mathbb{C}^2 , then the surface $H_1 = \overline{H}_1 \cap (\mathbb{C} \times \mathbb{C}^2)$ is given by equation

$$\frac{d!}{(d-4)!}ux^d + d^2(d-1)^2x^{d-2} - 2(d-1)(d-2)u^2x^2 + 2d(d-1)u = 0. \quad (13)$$

Since the polynomial in equation (13) depends only on the variables x and u , the surface H_1 is isomorphic to the product $E \times \mathbb{C}^1$, where E is a curve in \mathbb{C}^2 given by equation (13).

Let us show that the polynomial $H(u, x)$ in the left side of (13) is irreducible in the ring $\mathbb{C}[u, x]$. Indeed, assume that $H(u, x) = H_1(u, x)H_2(u, x)$. Then $H_1(u, x) = A_1(x)u + A_2(x)$ and $H_2(u, x) = A_3(x)u + A_4(x)$, since $H(u, x)$ is a polynomial of degree two in variable u and the polynomials $2(d-1)(d-2)x^2$ and $\frac{d!}{(d-4)!}x^d + 2d(d-1)$ are coprime. Therefore we have

$$A_1(x)A_3(x) = -2(d-1)(d-2)x^2, \quad A_2(x)A_4(x) = d^2(d-1)^2x^{d-2}, \quad (14)$$

$$A_1(x)A_4(x) + A_2(x)A_3(x) = \frac{d!}{(d-4)!}x^d + 2d(d-1). \quad (15)$$

It follows from (14) that $A_1(x) = b_1x^{k_1}$, and $A_3(x) = b_3x^{2-k_1}$, where $0 \leq k_1 \leq 2$ and $b_1b_3 = -2(d-1)(d-2) \in \mathbb{C}$. Similarly, $A_2(x) = b_2x^{k_2}$ and $A_4(x) = b_4x^{d-2-k_2}$, where $0 \leq k_2 \leq d-2$ and $b_2b_4 = d^2(d-1)^2 \in \mathbb{C}$. Therefore

$$b_1b_2b_3b_4 = -2d^2(d-1)^3(d-2). \quad (16)$$

It follows from (15) that

$$b_1b_4x^{d+k_1-k_2-2} + b_2b_3x^{k_2+2-k_1} = \frac{d!}{(d-4)!}x^d + 2d(d-1)$$

and therefore

$$b_1b_4 = \frac{d!}{(d-4)!} \text{ and } b_2b_3 = 2d(d-1) \text{ or } b_1b_4 = 2d(d-1) \text{ and } b_2b_3 = \frac{d!}{(d-4)!},$$

but in both cases we have

$$b_1b_2b_3b_4 = 2\frac{d!}{(d-4)!}d(d-1) = 2d^2(d-1)^2(d-2)(d-3). \quad (17)$$

It follows from (16) and (17) that we have the equality

$$2d^2(d-1)^2(d-2)(d-3) = 2d^2(d-1)^2(d-2)(d-3),$$

i.e., $d = 0, 1$ or 2 , but, by assumption, $d \geq 3$ and therefore $H(u, x)$ is an irreducible polynomial.

Denote by S the union of the set of critical values of the restriction $p : E \rightarrow \mathbb{C} \simeq L$ of the projection $\text{pr} : (u, x) \mapsto (u)$ to the irreducible curve E and the set of the images under the projection of the intersection points of E and the curve given by $x^d + ux^2 + 1 = 0$. Note that S is a finite set. Let $S = \{0, u_1, \dots, u_t\}$. Then for $\varepsilon \ll 1$ such that $\varepsilon < \min u_s$, where minimum is taken over all $u_s \in S \setminus \{0\}$, and for each fixed non-zero value u_0 of u , $|u_0| < \varepsilon$, the set $p^{-1}(u_0) = \{(u_0, x_1(u_0)), \dots, (u_0, x_d(u_0))\}$ consists of d different points such that two of these points, say $(u_0, x_{d-1}(u_0))$ and $(u_0, x_d(u_0))$, lie very far from the point $(0, 0)$ and the other $d-2$ points $(u_0, x_1(u_0)), \dots, (u_0, x_{d-2}(u_0))$ are very close to the point $(0, 0)$, since the closure of the line $\{u = 0\}$ meets the closure of the curve E at infinity with multiplicity two and at the point $(0, 0)$ with multiplicity $d-2$. Therefore

$$(\cup_{l=1}^d \mathcal{U}_{1,l}) \cap h_d^{-1}(u_0) = \{(u_0, x_s(u_0), y_l(u_0, x_s(u_0)))\}_{1 \leq s \leq d-2, 1 \leq l \leq d},$$

where $y_l(u_0, x_s(u_0))$, $l = 1, \dots, d$, are the roots of the equation

$$x_s^d(u_0) + y^d + 1 + u_0 x_s^2(u_0) = 0,$$

and hence the intersection $\mathcal{U}_{1,1} \cap h_d^{-1}(u_0)$ consists of $d-2$ different points for each $u_0 \in L_\varepsilon \setminus \{0\}$. Note also that $\mathcal{U}_{1,1} \cap h_d^{-1}(0)$ is the single point $(0, \bar{z}_{1,1})$. Therefore to prove that $\mathcal{U}_{1,1}$ is irreducible, it suffices to show that $\mathcal{U}_{1,1} \cap h_d^{-1}(L_\varepsilon)$ is a smooth curve, since otherwise the curve $\mathcal{U}_{1,1} \cap h_d^{-1}(L_\varepsilon)$ is the union of several components lying in different irreducible components of $\mathcal{U}_{1,1}$ and meeting at the point $(0, \bar{z}_{1,1})$ which must be the singular point of $\mathcal{U}_{1,1} \cap h_d^{-1}(L_\varepsilon)$.

To prove the smoothness of $\mathcal{U}_{1,1} \cap h_d^{-1}(L_\varepsilon)$ at $(0, \bar{z}_{1,1})$ note that in non-homogeneous coordinates $(u, x, y_1 = y - \mu_1)$ the curve $\mathcal{U}_{1,1} \cap h_d^{-1}(L_\varepsilon)$ is the complete intersection of two surfaces given by equation (13) and the equation $x^d + (y_1 + \mu_1)^d + 1 + ux^2 = 0$. It is easy to check that these two surfaces are non-singular at the point $(u, x, y_1) = (0, 0, 0)$ and meet transversally at this point. \square

Corollary 1. *Let $\mathcal{U}_{j,l} \subset \mathcal{U}_\varepsilon$ be the same as in Claim 4. Then $\mathcal{U}_{j,l} \setminus h_d^{-1}(\mathcal{B}_d)$ is a connected smooth variety.*

1.5. Transitivity of the actions of the groups \mathcal{G}_d . In notation used in subsection 1.4, let the base point \bar{a}_0 of the fundamental group $\pi_1(\mathbb{P}^{K_d} \setminus \mathcal{B}_d, \bar{a}_0)$ lie in the neighbourhood U_ε , $\varepsilon \ll 1$. Then the set $I_{\bar{a}_0} = h_d^{-1}(\bar{a})$ naturally splits into the union of $3d$ subsets, $I_{\bar{a}_0} = \bigsqcup_{j=1}^3 \bigsqcup_{l=1}^d I_{j,l}$, where $I_{j,l} = I_{\bar{a}} \cap \mathcal{U}_{j,l} = \{p_{j,l,1}, \dots, p_{j,l,d-2}\}$.

Claim 5. *The group $\mathcal{G}_d = \text{Im } h_{d*} \subset \mathbb{S}_{3d(d-2)}$ acts transitively on the set $I_{\bar{a}_0}$.*

Proof. In the beginning, let us show that the group \mathcal{G}_d acts transitively on each subset $I_{j,l}$. Indeed, by Corollary 1, for each pair (j, l) the variety $\mathcal{U}_{j,l} \setminus h_d^{-1}(\mathcal{B}_d)$ is connected. Therefore any two points $p_{j,l,m_1}, p_{j,l,m_2} \in I_{j,l}$ can be connected by a

smooth path $\gamma \subset \mathcal{U}_{j,l} \setminus h_d^{-1}(\mathcal{B}_d)$. Then the image $g = h_{d*}(\bar{\gamma}) \in \mathbb{S}_{3d(d-2)}$ of the element $\bar{\gamma} \in \pi_1(\mathbb{P}^{N_d} \setminus \mathcal{B}_d, \bar{a})$ represented by the loop $h_d(\gamma)$ sends the point p_{j,l,m_1} to p_{j,l,m_2} .

Now to complete the proof of Claim 5, it suffices to show that for each pair (j, l) the point $p_{1,1,1} \in I_{1,1}$ can be connected with some point $p_{j,l,m} \in I_{j,l}$ by a smooth path $l \subset \mathcal{I}_d \setminus \mathcal{B}_d$. For this let us consider an element $g_1 \in G_d \subset PGL(3, \mathbb{C})$ such that $g_1(\mathcal{U}_{1,1}) = \mathcal{U}_{j,l}$, where the group G_d was introduced in Subsection 1.4. The group $PGL(3, \mathbb{C})$ is connected. Therefore we can find a smooth path $g_t \subset PGL(3, \mathbb{C})$, $t \in [0, 1]$, connecting the elements $g_0 = Id$ and g_1 in $PGL(3, \mathbb{C})$. Then the path $g_t(p_{1,1,1})$ lies in $\mathcal{I}_d \setminus h_d^{-1}(\mathcal{B}_d)$, since for each $t \in [0, 1]$ the curve $g_t(C_{\bar{a}})$ is smooth and it has not multiple inflection points, and the point $g_t(p_{1,1,1})$ is an inflection point of the curve $g_t(C_{\bar{a}})$. Note also that the path $g_t(p_{1,1,1})$ connects the point $p_{1,1,1}$ with some point $g_1(p_{1,1,1}) \in \mathcal{U}_{j,l}$. As above, by Corollary 1, the point $g_1(p_{1,1,1})$ can be connected with any point lying in $I_{j,l}$ by a path in $\mathcal{U}_{j,l} \setminus h_d^{-1}(\mathcal{B}_d)$. \square

1.6. Behaviour of the covering h_d near a 2-tuple inflection point. Let p be a 2-tuple inflection point of a curve C of degree $d \geq 4$. Without loss of generality, we can assume that $p = (0, 0, 1)$ and the line $\{z_1 = 0\}$ is the tangent line to C at the point p . Then C is given by equation $F(z_1, z_2, z_3) = 0$, where $F(z_1, z_2, z_3)$ is a homogeneous polynomial of the form $z_2^4 R(z_2, z_3) + z_1 S(z_1, z_2, z_3)$ and where $R(z_1, z_3)$ is a homogeneous polynomial of degree $d - 4$ such that $R(0, 1) = 1$, and $S(z_1, z_2, z_3)$ is a homogeneous polynomial of degree $d - 1$ such that $S(0, 0, 1) = 1$.

Let $V \subset \mathbb{P}^{K_d}$ be a very small neighbourhood of the point c corresponding to the curve C and $\mathcal{V} = h_d^{-1}(V) \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$. Then \mathcal{V} is the disjoint union of two components, $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$, where \mathcal{V}_1 contains the point $(c, p) \in \mathbb{P}^{K_d} \times \mathbb{P}^2$. Obviously, the restriction of h_d to \mathcal{V}_1 has degree two, since p is a 2-tuple inflection point of C and therefore under small deformation of C the deformed curves, near the point p , have either two different inflection points or one 2-tuple inflection point.

Claim 6. *There is a smooth curve $E_1 \subset \mathcal{V}_1$ passing through the point (c, p) and such that the restriction $h_{d|E_1} : E_1 \rightarrow L_1$ of h_d to E_1 is ramified at (c, p) with multiplicity $\deg h_{d|E_1} = 2$.*

Proof. Consider the family of curves $C_v \subset \Delta \times \mathbb{P}^2 \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$ given by $F(z_1, z_2, z_3) + v z_2^2 z_3^{d-2} = 0$, where $\Delta = \{|v| < \varepsilon_1\}$ is the disk in \mathbb{C} of small radius ε_1 .

Consider the curve $E_1 = \mathcal{V}_1 \cap C_v \cap H_v$, where H_v is the Hessian of the family C_v . Let $x = z_1/z_3$ and $y = z_2/z_3$. Denote by $R'_j = \frac{\partial R}{\partial z_j}(x, 1)$, $S'_j = \frac{\partial S}{\partial z_j}(x, y, 1)$, $R''_{j,l} = \frac{\partial^2 R}{\partial z_j \partial z_l}(x, 1)$, and $R''_{j,l} = \frac{\partial^2 R}{\partial z_j \partial z_l}(x, y, 1)$. In the coordinates (v, x, y) the family C_v is given by

$$y^4 R(y, 1) + v y^2 + x S(x, y, 1) = 0 \quad (18)$$

and the Hessian H_v is given by

$$\begin{vmatrix} 2S'_1 + xS''_{1,1}, & S'_2 + xS''_{1,2}, & S'_3 + xS''_{1,3} \\ S'_2 + xS''_{1,2}, & 12y^2 R + 8y^3 R'_2 + y^4 R''_{2,2} + 2v + xS''_{2,2}, & 4y^3 R'_3 + y^4 R''_{2,3} + 2\alpha v y + xS''_{2,3} \\ S'_3 + xS''_{1,3}, & 4y^3 R'_3 + y^4 R''_{2,3} + 2\alpha v y + xS''_{2,3}, & y^4 R''_{3,3} + \beta v y^2 + xS''_{3,3} \end{vmatrix} = 0, \quad (19)$$

where $\alpha = d - 2$ and $\beta = (d - 2)(d - 3)$.

Let $S'_j(0, 0, 1) = s_j$ and $S''_{j,l}(0, 0, 1) = s_{j,l}$. By assumption, $S(0, 0, 1) = 1$. Therefore $S'_3(0, 0, 1) = d - 1$ and it is easy to see that the differentials at the point $(v, x, y) = (0, 0, 0)$ of the polynomials in the left side of equations (18) and (19) are equal respectively to $\bar{d}x$ and $(2s_2s_3s_{2,3} - s_2^2s_{3,3})\bar{d}x - 2(d - 1)^2\bar{d}v$ (here we denote by $\bar{d}f$ the differential of function f in order not to confuse with degree d of the curve C). Therefore the surfaces C_v and H_v are nonsingular at the point $(0, 0, 0)$ and meet transversally at this point, since these differentials are linear independent. It follows from this that $E_1 = \mathcal{V}_1 \cap C_v \cap H_v$ is a smooth curve and the differential of $h_{d|E_1} : E_1 \rightarrow \Delta$ vanishes at the point $(0, 0, 0)$. Therefore $\deg h_{d|E_1} = 2$ since if a holomorphic surjective map of a smooth curve has degree one, then its differential vanishes nowhere. \square

Corollary 2. *Let C be a smooth curve of degree $d \geq 4$ having $3d(d - 2) - 1$ inflection points. Then, in notations used in the proof of Claim 6, $h_d^{-1}(\Delta) \cap \mathcal{V}$ is the disjoint union of $3d(d - 2) - 1$ smooth curves, $h_d^{-1}(\Delta) \cap \mathcal{V} = \bigsqcup_{j=1}^{3d(d-2)-1} E_j$, where $h_{d|E_j} : E_j \rightarrow \Delta$ is a bi-holomorphic map for $j = 2, \dots, 3d(d - 2) - 1$ and $h_{d|E_1}$ is a two-sheeted covering branched at the point $\{v = 0\} \in L_1$.*

Consider a point $(\bar{a}, \bar{z}) \in h_d^{-1}(\mathfrak{M}_d)$, where $\mathfrak{M}_d \subset \mathbb{P}^{K_d}$ is the variety defined in Subsection 1.2. Lemma 2 and Claim 6 imply

Proposition 2. *Let $\bar{a}_0 \in \mathfrak{M}_d$ and \bar{z}_0 be a 2-tuple inflection point of the curve $C_{\bar{a}_0}$. Then (\bar{a}_0, \bar{z}_0) is a smooth point of the variety \mathcal{I}_d .*

Corollary 2 and Lemma 1 imply

Claim 7. *For $d \geq 4$ the group $\mathcal{G}_d \subset \mathbb{S}_{3d(d-2)}$ contains a transposition.*

1.7. Case $d = 3$. Consider the *Hesse pencil*, that is, the one-dimensional linear system of plane cubic curves given by

$$C_{(t_1, t_2)} : t_1(z_1^3 + z_2^3 + z_3^3) + t_2z_1z_2z_3 = 0, \quad (t_1, t_2) \in \mathbb{P}^1, \quad (20)$$

We call the surface $\mathcal{H} \subset L \times \mathbb{P}^2 \subset \mathbb{P}^{K_3} \times \mathbb{P}^2$ given in $L \times \mathbb{P}^2$ by equation (20) the *body* of the Hesse pencil, where $L \simeq \mathbb{P}^1$ and $K_3 = 9$.

It is easy to see that \mathcal{H} is a smooth surface and the restriction $\sigma : \mathcal{H} \rightarrow \mathbb{P}^2$ of pr_2 to \mathcal{H} is the composition of nine σ -processes of \mathbb{P}^2 with centers at the base points of the Hesse pencil. Let $E_{q_j} = \sigma^{-1}(q_j)$, $j = 1, \dots, 9$, be the exceptional curve of σ over the base point $q_j \in \mathbb{P}^2$ of the pencil. The curves E_j are sections of the projection $\text{pr}_1 : L \times \mathbb{P}^2 \rightarrow L$.

It is well-known (see, for example, [1]) that the base points of the Hesse pencil are the inflection points of each smooth member of the pencil. The Hesse pencil has four degenerate members, $C_{(0,1)}$, $C_{(1,-3)}$, $C_{(1,-3e^{2\pi i/3})}$, and $C_{(1,-3e^{4\pi i/3})}$. Each of the degenerate members is the union of three lines. Therefore

$$\mathcal{I}_3 \cap \mathcal{H} = C_{(0,1)} \cup C_{(1,-3)} \cup C_{(1,-3e^{2\pi i/3})} \cup C_{(1,-3e^{4\pi i/3})} \cup (\cup_{j=1}^9 E_j).$$

The group $Hes \subset PGL(3, \mathbb{C})$ of the projective transformations leaving invariant the set of the inflection points of the Fermat curve $F_3 = C_{(1,0)}$ is well investigated (see, for example, [1]). The order of Hes is equal to 216 and the action of Hes on the 9 inflection points of F_3 defines an imbedding $Hes \subset S_9$ such that Hes is a 2-transitive subgroup of S_9 . The orbit of the Fermat curve F_3 under the action of Hes consists of four members of the Hesse pencil: $F = C_{(1,0)}, C_{(1,6)}, C_{(1,6e^{2\pi i/3})}, C_{(1,6e^{4\pi i/3})}$. We choose three continuous paths l_j , $j = 0, 1, 2$, in $L \setminus \{(0, 1), (1, -3), (1, -3e^{2\pi i/3}), (1, -3e^{4\pi i/3})\}$ connecting the points $(1, 6e^{2j\pi i/3})$ with the point $(1, 0)$.

It is well-known ([3], [4]) that the set of nine inflection points of a plane cubic curve is a projectively rigid set, that is, for each two smooth plane cubic curves C_1 and C_2 there is a projective transformation of the plane sending the set of the inflection points of C_1 onto the set of the inflection points of C_2 . Therefore there is an imbedding $\varphi : PGL(3, \mathbb{C}) \rightarrow (\mathbb{P}^2)^9$ given for $\tau \in PGL(3, \mathbb{C})$ by

$$\varphi : \tau \mapsto (\tau(q_1), \dots, \tau(q_9)) \in (\mathbb{P}^2)^9,$$

where $\{q_1, \dots, q_9\} \subset \mathbb{P}^2$ is the set of the inflection points of the Fermat curve F_3 . (Note that φ depends on the numbering of the inflection points of the Fermat curve F_3 , that is, there are $\frac{9!}{216}$ such imbeddings.) Denote by $\mathcal{P} = \varphi(PGL(3, \mathbb{C})) \subset (\mathbb{P}^2)^9$.

Claim 8. *We have $\mathcal{G}_3 = Hes \subset S_9$.*

Proof. Consider the homomorphism $h_{3*} : \pi_1(\mathbb{P}^{K_3} \setminus \mathcal{S}_3, f) \rightarrow S_9$, where $f = (1, 0) \in L \subset \mathbb{P}^{K_3}$ is the point corresponding to the Fermat curve F_3 . Then $\mathcal{G}_3 = h_{3*}(\pi_1(\mathbb{P}^{K_3} \setminus \mathcal{S}_3, f))$ acts on the set $h_3^{-1}(f) = \{q_1, \dots, q_9\}$.

Let us show that $Hes \subseteq \mathcal{G}_3$. For this, consider an element $g_1 \in Hes \subset PGL(3, \mathbb{C})$. Since $PGL(3, \mathbb{C})$ is a connected variety, we can choose a continuous path $g_t \subset PGL(3, \mathbb{C})$, $0 \leq t \leq 1$, connecting $g_0 = Id$ with g_1 . The path g_t defines a continuous path $l(t) \subset \mathbb{P}^{K_3} \setminus \mathcal{S}_3$ such that $C_{l(t)} = g_t(F_3)$. We have $C_{l(1)}$ is a member of the Hesse pencil, since $g_1 \in Hes$. Denote by $\Gamma \subset \mathbb{P}^{K_3} \setminus \mathcal{S}_3$ the path $l(t)$ if $C_{l(1)} = F_3$ and $l(t) \cup l_j$ if $C_{l(1)} = C_{(1, 6e^{2j\pi i/3})}$. Then the loop Γ represents an element $\gamma \in \pi_1(\mathbb{P}^{K_3} \setminus \mathcal{S}_3, f)$ and it is easy to see that the action of $h_{3*}(\gamma)$ on $h_3^{-1}(f) = \{q_1, \dots, q_9\}$ is the same as the action of g_1 .

Let us show that $\mathcal{G}_3 \subseteq Hes$. For this, consider an element $g \in \mathcal{G}_3$ and a continuous loop $\Gamma(t) \subset \mathbb{P}^{K_3} \setminus \mathcal{S}_3$ starting and ending at f and representing an element $\gamma \in \pi_1(\mathbb{P}^{K_3} \setminus \mathcal{S}_3, f)$ such that $h_{3*}(\gamma) = g$. We lift the loop $\Gamma(t)$ to \mathcal{I}_3 and this lift consists of 9 continuous paths $\Gamma_1(t), \dots, \Gamma_9(t) \subset \mathcal{I}_3$ starting and ending at the points of $h_3^{-1}(f)$. So, we obtain 9 continuous paths $pr_2(\Gamma_1(t)), \dots, pr_2(\Gamma_9(t)) \subset \mathbb{P}^2$. If we number the paths $\Gamma_j(t)$ so that $\Gamma_j(0) = q_j$, then $(pr_2(\Gamma_1(t)), \dots, pr_2(\Gamma_9(t)))$ is a continuous path in \mathcal{P} , since $\{pr_2(\Gamma_1(t)), \dots, pr_2(\Gamma_9(t))\}$ is the set of the inflection points of smooth plane cubic curves. Therefore there is an element $\tau \in PGL(3, \mathbb{C})$ such that $\tau(q_j) = pr_2(\Gamma_j(1)) \in h_3^{-1}(f)$ for $j = 1, \dots, 9$, that is, $g = \tau \in Hes$. \square

1.8. **Case $d = 4$.** Consider the Klein curve $Kl \subset \mathbb{P}^2$ given by

$$z_1^3 z_2 + z_1^3 z_3 + z_1 z_2^3 + z_2^3 z_3 + z_1 z_3^3 + z_2 z_3^3 = 0.$$

It is well-known (see, for example, [2]) that the automorphism group $Aut(Kl)$ of Kl have the following properties. The order of $Aut(Kl)$ is equal to 168 and $Aut(Kl) \simeq PSL(2, \mathbb{Z}_7)$, the group $Aut(Kl)$ can be represented as a subgroup of $PGL(3, \mathbb{C})$ leaving invariant the curve Kl and the set of inflection points is an orbit under the action $Aut(Kl)$, the order of the stabilizer of each inflection point is equal to 7. In particular, the action of $Aut(Kl)$ on the set of the inflection points of Kl is transitive.

Claim 9. *There is an imbedding $Aut(Kl) \subset \mathcal{G}_4 \subset \mathbb{S}_{24}$ such that $Aut(Kl)$ is a transitive subgroup of \mathbb{S}_{24} .*

Proof. Let $\bar{a}_0 \in \mathbb{P}^{K_4}$ be the point corresponding to the curve Kl and g_1 an element of $Aut(Kl) \subset PGL(3, \mathbb{C})$. Since $PGL(3, \mathbb{C})$ is connected, there is a continuous path $g_t \subset PGL(3, \mathbb{C})$, $t \in [0, 1]$, connecting $g_0 = Id$ and g_1 . Then the loop $\Gamma \subset \mathbb{P}^{K_4} \setminus \mathcal{B}_4$ given by $g_t(\bar{a}_0)$, $t \in [0, 1]$, defines an element $\gamma \in \pi_1(\mathbb{P}^{K_4} \setminus \mathcal{B}_4, k)$ such that the action of $h_{4*}(\gamma) \in \mathbb{S}_{24}$ on the set $h_4^{-1}(\bar{a}_0)$ of the inflection points of Kl coincides with the action of $g_1 \in Aut(Kl)$. \square

By Claim 3 and 9, the group $\mathcal{G}_4 \subset \mathbb{S}_{24}$ has the following properties:

- (1) \mathcal{G}_4 contains a subgroup $Aut(Kl)$ which acts transitively on the set $I_{24} = \{1, 2, \dots, 23, 24\}$;
- (2) there are a partition $I_{24} = J_1 \sqcup J_2$, $|J_1| = 9$, $|J_2| = 15$, and a quasi-imbedding $\mathcal{G}_3 \prec \mathcal{G}_4$ such that J_1 is invariant under the action of $\tilde{\mathcal{G}}_3 \subset \mathcal{G}_4$ (see subsection 1.3) and the action of $\tilde{\mathcal{G}}_3$ on J_1 is 2-transitive;
- (3) the group \mathcal{G}_4 contains a transposition.

Claim 10. *Properties (1) – (3) imply $\mathcal{G}_4 = \mathbb{S}_{24}$.*

Proof. We say that a subset $J \subset I_{24}$ is 2-transitive (with respect to the action of \mathcal{G}_4) if for each two pairs $\{j_1, j_2\} \subset J$ and $\{j_3, j_4\} \subset J$ there is an element $g \in \mathcal{G}_4$ such that $g(\{j_1, j_2\}) = \{j_3, j_4\}$.

Denote by $\tilde{J} \subset I_{24}$ a 2-transitive subset of maximal cardinality. Obviously, if $J \subset I_{24}$ is a 2-transitive subset then for each $g \in \mathcal{G}_4$ the set $g(J)$ is also 2-transitive, and if J_1 and J_2 are 2-transitive subsets such that the cardinality $|J_1 \cap J_2| \geq 2$, then $J_1 \cup J_2$ is also 2-transitive. Therefore it is easy to see that there are two possibilities: either $\tilde{J} = I_{24}$, or $|\tilde{J}| = 12$, since, by property (2), the cardinality $|\tilde{J}| \geq 9$ and, by property (1), \mathcal{G}_4 acts transitively on I_{24} .

Let us show that the second case is impossible. Indeed, in this case it follows from transitivity of the action of \mathcal{G}_4 that for each $g \in \mathcal{G}_4$ either $g(\tilde{J}) = \tilde{J}$, or $g(\tilde{J}) = I_{24} \setminus \tilde{J}$. Therefore the action of \mathcal{G}_4 on I_{24} induces an action on the set $\{\tilde{J}, I_{24} \setminus \tilde{J}\}$ of cardinality two, that is, there is an epimorphism $\varphi : \mathcal{G}_4 \rightarrow \mathbb{Z}_2$. But, in this case the restriction $\varphi|_{Aut(Kl)} : Aut(Kl) \rightarrow \mathbb{Z}_2$ is also an epimorphism, since $Aut(Kl)$ acts transitively

on the set I_{24} . On the other hand, $\text{Aut}(Kl) \simeq \text{PSL}(2, \mathbb{Z}_7)$ is a simple group and therefore $\varphi|_{\text{Aut}(Kl)}$ must be trivial homomorphism.

Now, to complete the proof of Claim 10, it suffices to apply property (3), since \mathcal{G}_4 acts 2-transitively on I_{24} and therefore the all transpositions of \mathbb{S}_{24} are contained in \mathcal{G}_4 . \square

1.9. The end of the proof of Theorem 1. To complete the proof of Theorem 1 we need in the following

Lemma 3. *Let G be a subgroup of the symmetric group \mathbb{S}_m acting on a finite set M of cardinality m . Assume that*

- (i) G acts transitively on M ;
- (ii) there are a subgroup G_1 of G and a subset M_1 of M such that
 - (ii)₁ M_1 is invariant under the action of the group G_1 ,
 - (ii)₂ $2m_1 \geq m + 2$, where m_1 is the cardinality of the set M_1 ,
 - (ii)₃ the action of the group G_1 on M_1 is 2-transitive;
- (iii) the group G contains a transposition.

Then $G = \mathbb{S}_m$.

Proof. By (ii)₃, for each element $g \in G$ the subgroup gG_1g^{-1} of G acts 2-transitively on $g(M_1)$ and by (ii)₂, the group G acts 2-transitively on $M_1 \cup g(M_1)$, since there are at least two elements in the intersection of $M_1 \cap g(M_1)$. It follows from (i) that for each element $x \in M$ there is an element $g \in G$ such that $x \in g(M_1)$. Therefore G acts 2-transitively on M and hence applying (iii) the group $G \subset \mathbb{S}_m$ contains all transpositions. \square

Now, applying induction on d , Theorem 1 follows from Claims 3, 5, 7, 8, 10 and Lemma 3, since

$$2[3d(d-2)] \geq [3(d+1)(d-1)] + 2$$

for $d \geq 4$.

2. BEHAVIOUR OF THE COVERING h_d NEAR A NODE OF A NODAL CURVE

2.1. On the subset of \mathcal{S}_d consisting of the points corresponding to the nodal curves. Denote by \mathcal{N}_d a subvariety of \mathcal{S}_d consisting of the points $\bar{a} \in \mathcal{S}_d$ such that the set of singular points of the curves $C_{\bar{a}}$ consist of the only one ordinary node. The following Proposition is well-known.

Proposition 3. *The variety \mathcal{S}_d is an irreducible hypersurface in \mathbb{P}^{K_d} for each $d \geq 3$. The variety \mathcal{N}_d is a non-empty Zariski open subset of \mathcal{S}_d .*

Proof of this proposition is similar to the proof of Proposition 1 and therefore it will be omitted. \square

Claim 11. *The variety $\mathcal{N}_d \subset \mathbb{P}^{K_d}$ is smooth.*

Proof. Consider a point $\bar{a}_0 \in \mathcal{N}_d$. Without loss of generality, we can assume that $\bar{z}_0 = (0, 0, 1)$ is the singular point of $C_{\bar{a}_0}$ and $C_{\bar{a}_0}$ is given by equation $F(\bar{a}_0, \bar{z}) = 0$, where

$$F(\bar{a}_0, \bar{z}) = z_1 z_2 z_3^{d-2} + R(z_1, z_2, z_3),$$

and where $R(z_1, z_2, z_3)$ is a homogeneous polynomial of degree d such that it, as a polynomial in variable z_3 , has degree $\leq d - 3$. In particular, the coordinates $\alpha_{0,0,d}$, $\alpha_{1,0,d-1}$, $\alpha_{0,1,d-1}$, $\alpha_{2,0,d-2}$, $\alpha_{0,2,d-2}$ in \bar{a}_0 are equal to zero. Let us show that the tangent space $T_{\bar{a}_0} \mathcal{S}_d$ to \mathcal{S}_d at \bar{a}_0 is given by equation $a_{0,0,d} = 0$.

The variety $\mathcal{C}_d \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$ at the point (\bar{a}_0, \bar{z}_0) in non-homogeneous coordinates is given by

$$a_{0,0,d} + a_{1,0,d-1}x + a_{0,1,d-1}y + xy + a_{2,0,d-2}x^2 + a_{0,2,d-2}y^2 + \tilde{R}(x, y, 1) = 0. \quad (21)$$

It is easy to see that \mathcal{C}_d is non-singular at (\bar{a}_0, \bar{z}_0) and the elements of the set of variables $\{x, y\} \cup \{a_{k,m,n}\}_{k+m+n=d} \setminus \{a_{0,0,d}, a_{1,1,d-2}\}$ are local coordinates at (\bar{a}_0, \bar{z}_0) . Consider in \mathcal{C}_d a subvariety $Sing_d$ given by

$$a_{1,0,d-1} + y + 2a_{2,0,d-2}x + \tilde{R}'_x(x, y, 1) = 0 \quad (22)$$

and

$$a_{0,1,d-1} + x + 2a_{0,2,d-2}y + \tilde{R}'_y(x, y, 1) = 0. \quad (23)$$

Obviously, in a neighbourhood of \bar{a}_0 the variety \mathcal{S}_d is the image of $Sing_d$ under the morphism \tilde{h}_d . It follows from (21) – (23) that $Sing_d$ is non-singular at (\bar{a}_0, \bar{z}_0) , the elements of the set of variables $\{a_{k,m,n}\}_{k+m+n=d} \setminus \{a_{0,0,d}, a_{1,1,d-2}\}$ are local coordinates at (\bar{a}_0, \bar{z}_0) and $\tilde{h}_d : Sing_d \rightarrow \mathcal{S}_d$ is given in these coordinates by

$$a_{0,0,d} = a_{1,0,d-1}^2 A_1(a_{k,m,n}) + a_{1,0,d-1} a_{0,1,d-1} A_2(a_{k,m,n}) + a_{0,1,d-1}^2 A_3(a_{k,m,n}),$$

where $A_j(a_{k,m,n})$, $j = 1, 2, 3$, are power serieses in variables

$$\{a_{k,m,n}\}_{k+m+n=d} \setminus \{a_{0,0,d}, a_{1,1,d-2}\}. \quad \square$$

Let \bar{z}_0 be an ordinary node of $C_{\bar{a}_0}$. In [5], it was shown that $(C, H_C)_{\bar{z}_0} = 6$ if \bar{z}_0 is not an inflection point of each of the branches of $C_{\bar{a}_0}$ at \bar{z}_0 . Therefore the local degree of h_d at the point (\bar{a}, \bar{z}_0) is equal to $\deg_{(\bar{a}_0, \bar{z}_0)} h_d = 6$.

2.2. On the local monodromy groups of h_d at the points corresponding to the nodal curves. Denote $\mathfrak{N}_d = \mathcal{N}_d \setminus \mathcal{M}_d$.

Proposition 4. *Let $\bar{a}_0 \in \mathfrak{N}_d$ and \bar{z}_0 be the singular point of $C_{\bar{z}_0}$. Then the local monodromy group $\mathcal{G}_{d, \bar{a}_0} \subset \mathbb{S}_{3d(d-2)}$ at the point \bar{a}_0 is a cyclic group of order 3 and it is generated by the product of two disjoint cycles of length 3.*

Proof. Without loss of generality we can assume that $C_{\bar{a}_0}$ is given by equation $F(\bar{a}_0, \bar{z}) = 0$, where

$$F(\bar{a}_0, \bar{z}) = z_1 z_2 z_3^{d-2} + z_3^{d-3} \sum_{j+k=3} \alpha_{j,k,d-3} z_1^j z_2^k + R(z_1, z_2, z_3)$$

and where R is a polynomial of degree $\leq d-4$ in the variable z_3 .

Note that $a_{3,0,d-3} \neq 0$ and $a_{0,3,d-3} \neq 0$ if $\bar{a}_0 \in \mathfrak{N}_d$. Indeed, if, for example, $a_{3,0,d-3} = 0$ then it is easy to see that the line L given by $t\bar{u} + \bar{a}_0$, where $t \in \mathbb{C}$ and in \bar{u} all coordinates except the coordinate $u_{0,1,d-1}$ are equal to zero and $u_{0,1,d-1} = 1$, lies in \mathcal{M}_d .

Consider a one-parametric family $C_{\bar{a}_t}$ given by equation

$$F(\bar{a}_0, \bar{z}) + tz_3^d = 0$$

and its projection $\text{pr}_1(C_{\bar{a}_t}) = L = \{\bar{a}_t = \bar{a}_0 + t\bar{v}\} \subset \mathbb{P}^{K_d}$, where in \bar{v} all coordinates except the coordinate $v_{0,0,d}$ are equal to zero and $v_{0,0,d} = 1$. By Claim 11, L meets \mathcal{S}_d transversally at \bar{a}_0 .

In non-homogeneous coordinates $x = \frac{z_1}{z_3}, y = \frac{z_2}{z_3}$ we have $\bar{z}_0 = (0, 0)$, the family $C_{\bar{a}_t}$ in $L \times \mathbb{C}^2 \subset \mathbb{P}^{K_d} \times \mathbb{P}^2$ is given by equation

$$t + xy + \sum_{i+j=3} a_{i,j,d-3} x^i y^j + \text{terms of higher degree} = 0, \quad (24)$$

and everybody can easily check that its Hessian $H_{C_{\bar{a}_t}}$ is given by equation

$$\begin{aligned} & 2(d-2)^2(xy - 3a_{3,0,d-3}x^3 + a_{2,1,d-3}x^2y + a_{1,2,d-3}xy^2 - 3a_{0,3,d-3}y^3) + \\ & d(d-1)(1 + 4a_{2,1,d-3}x + 4a_{1,2,d-3}y)t + r_1(x, y) + tr_2(x, y) = 0, \end{aligned} \quad (25)$$

where $r_1(x, y) = \sum_{i+j \geq 4} b_{i,j} x^i y^j$ and $r_2(x, y) = \sum_{i+j \geq 2} c_{i,j} x^i y^j$ are some polynomials.

Consider the curve $Z = h_d^{-1}(L) = C_{\bar{a}_t} \cap H_{C_{\bar{a}_t}} \subset X$, where X is a surface in \mathbb{C}^3 given by equation (24). It is easy to see that X is isomorphic to $\mathbb{C}^2 = \text{Spec } \mathbb{C}[x, y]$ and Z in X is given by equation

$$\begin{aligned} & (d^2 - 7d + 8)xy - 6(d-2)^2(a_{3,0,d-3}x^3 + a_{0,3,d-3}y^3) - \\ & 2(d^2 + 2d - 4)(a_{2,1,d-3}x^2y + a_{1,2,d-3}xy^2) + \text{terms of higher degree} = 0, \end{aligned} \quad (26)$$

since

$$t = -(xy + \sum_{i+j=3} a_{i,j,d-3} x^i y^j + \text{terms of higher degree}). \quad (27)$$

It follows from (26) that Z has a node at the point $\mathbf{p} = (\bar{a}_0, \bar{z}_0)$. To resolve this point, consider the σ -process $\sigma : \tilde{X} \rightarrow X$ with center at \mathbf{p} . The surface \tilde{X} is covered by two open neighbourhoods isomorphic to \mathbb{C}^2 , $\tilde{X} = U_1 \cup U_2$. The coordinates in U_j , $j = 1, 2$, are x_j, y_j and $\sigma|_{U_1}$ is given by $x = x_1$ and $y = x_1 y_1$, and $\sigma|_{U_2}$ is given by $x = x_2 y_2$ and $y = y_2$. Therefore the proper inverse image $\sigma^{-1}(Z) \cap U_1$ is given by equation

$$\begin{aligned} & (d^2 - 7d + 8)y_1 - 6(d-2)^2(a_{3,0,d-3}x_1 + a_{0,3,d-3}x_1 y_1^3) - \\ & 2(d^2 + 2d - 4)(a_{2,1,d-3}x_1 y_1 + a_{1,2,d-3}x_1 y_1^2) + \text{terms of higher degree} = 0, \end{aligned} \quad (28)$$

and therefore the curve $\tilde{Z} = \sigma^{-1}(Z)$ is non-singular at the point $\mathbf{p}_1 = \tilde{Z} \cap U_1 \cap E$, where E is the exceptional divisor of σ .

Since $a_{3,0,d-3} \neq 0$, the coordinate x_1 is a local parameter in \tilde{L} at the point \mathbf{p}_1 and

$$y_1 = \frac{6(d-2)^2 a_{3,0,d-3}}{d^2 - 7d + 8} x_1 + \sum_{j=2}^{\infty} b_j x_1^j.$$

It follows from (27) that

$$t = \frac{x_1^2 y_1 + a_{3,0,d-3} x_1^3}{\left(\frac{6(d-2)^2}{d^2 - 7d + 8} + 1\right) a_{3,0,d-3} x_1^3} + \text{terms of higher degree} =$$

Therefore the covering $h_d \circ \sigma : \tilde{Z} \rightarrow L$ is ramified at \mathbf{p}_1 with multiplicity three.

Similar calculations (which will be omitted) show that the covering $h_d \circ \sigma : \tilde{Z} \rightarrow L$ also is ramified at $\mathbf{p}_2 = \tilde{Z} \cap U_2 \cap E$ with multiplicity three, since $a_{0,3,d-3} \neq 0$. \square

Let $\nu_d : \mathfrak{J}_d \rightarrow \mathcal{I}_d$ be the normalization of the variety \mathcal{I}_d and $\bar{h}_d = h_d \circ \nu_d : \mathfrak{J}_d \rightarrow \mathbb{P}^{K_d}$. The following Proposition is an easy corollary of Lemma 1, Claim 1, and Proposition 4.

Proposition 5. *Let $\bar{a}_0 \in \mathfrak{N}_d$ and \bar{z}_0 is the singular point of the curve $C_{\bar{a}_0}$. Then*

- (i) *the variety \mathfrak{J}_d is smooth at the point $\mathbf{p} = (\bar{a}_0, \bar{z}_0)$;*
- (ii) *$\nu_d^{-1}(\mathbf{p}) = \{\mathbf{q}_1, \mathbf{q}_2\}$ consists of two points;*
- (iii) *\bar{h}_d is ramified along $\nu_d^{-1}(\mathfrak{N}_d)$ and the local degree $\deg_{\mathbf{q}_j} \bar{h}_d = 3$ for $j = 1, 2$.*

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